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Conformal Invariance and the Critical Behavior of the Triplet XY Quantum Chain

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We introduce and study the critical properties of the triplet XY quantum chain. This system is described in terms of three-spin interactions and is the generalization of the standard XY quantum chain. We show that this model, with periodic boundaries, has a local gauge invariance and can be described by the composition of two triplet Ising models, with general toroidal boundary conditions. From this composition the phase diagram as well the conformal anomaly and critical exponents are determined by exploring their relations with the mass gap amplitudes predicted by conformal invariance.

KEY WORDS: Ising models; finite-size scaling; multispin interactions; conformal invariance; XY model.

1. INTRODUCTION

Recently models with multispin interactions have received increasing attention. These models exhibit a rich variety of critical behavior; the classical examples in two dimensions are the eight-vertex model[(1)] and the Ashkin–Teller model.[(2)] Both models can be formulated as Ising models with two- and four-spin interactions and have a critical line with continuously varying critical exponents.

In a previous paper Alcaraz and Barber[(3)] introduced the triplet Ising model and showed that its critical behavior has the same richness as the Ashkin–Teller model. This model is defined by mixed two- and three-body interactions in the square lattice and is the anisotropic generalization of

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the Ising model with three-spin interactions proposed by Debierre and Turban.[4]

In this paper we study the anisotropic triplet XY quantum chain, defined in terms of three-spin interactions, which is the generalization of the standard XY quantum Hamiltonian, having only two-spin interactions. Instead of the $U(1)$ symmetry present in the standard XY model, the triplet generalization exhibits a local gauge symmetry which permits us to derive important consequences. In the next section we define the model and we also show that it can be decomposed into two alternate triplet XY models each having one-half of the three-spin interactions. In Section 3 we verify that these alternate models have a $Z(2)$ local gauge symmetry. Exploring this symmetry, we show in Section 4 that the last models are related to the triplet Ising model[3] with boundary conditions dependent on the gauge we choose for the alternate triplet models. Consequently, in order to study the triplet XY model, we have to analyze the anisotropic Ising model with several boundary conditions. The case of periodic boundary condition already has been analyzed[3] and in Section 5 we generalize these studies for several boundaries. The analysis was done by exploring the consequences of the conformal invariance of the critical infinite system in the spectra of the Hamiltonian in a finite-size strip. Several scaling dimensions, which are related to the critical exponents, are calculated. Finally, Section 6 closes with our conclusions and a summary of our results.

2. THE MODEL

The isotropic triplet XY quantum chain is defined on an L-site chain by the Hamiltonian

$$H_{XY}^{(3)}(0, \lambda) = -\sum_{i=0}^{L-1} (\sigma_i^X \sigma_{i+1}^X \sigma_{i+2}^X + \lambda \sigma_i^Y \sigma_{i+1}^Y \sigma_{i+2}^Y) \tag{2.1}$$

where σ^X and σ^Y are spin-1/2 Pauli matrices, λ plays the role of the temperature, and periodic boundary conditions are assumed. This Hamiltonian is the three-spin generalization of the standard XY model, which only has two-body interactions, and is defined by the following Hamiltonian:

$$H_{XY}^{(2)}(\lambda) = -\sum_{i=0}^{L-1} (\sigma_i^X \sigma_{i+1}^X + \lambda \sigma_i^Y \sigma_{i+1}^Y) \tag{2.2}$$

The model defined by (2.1) is the isotropic version of the more general anisotropic triplet Hamiltonian

$$
\begin{aligned}
H_{XY}^{(3)}(\alpha, \lambda) = -\Bigg\{ & \sum_{i=0}^{L/3-6} (\alpha \sigma_{3i}^X \sigma_{3i+1}^X \sigma_{3i+2}^X + \alpha \sigma_{3i+1}^X \sigma_{3i+2}^X \sigma_{3i+3}^X \\
& + \alpha \sigma_{3i+2}^X \sigma_{3i+3}^X \sigma_{3i+4}^X) + \alpha \sigma_{L-3}^X \sigma_{L-2}^X \sigma_{L-1}^X \\
& + \alpha \sigma_{L-2}^X \sigma_{L-1}^X \sigma_0^X + \sigma_{L-1}^X \sigma_0^X \sigma_1^X \\
& + \lambda \Bigg[\sum_{i=0}^{L/3-6} (\alpha \sigma_{3i}^Y \sigma_{3i+1}^Y \sigma_{3i+2}^Y + \alpha \sigma_{3i+1}^Y \sigma_{3i+2}^Y \sigma_{3i+3}^Y \\
& + \alpha \sigma_{3i+2}^Y \sigma_{3i+3}^Y \sigma_{3i+4}^Y) + \alpha \sigma_{L-3}^Y \sigma_{L-2}^Y \sigma_{L-1}^Y \\
& + \alpha \sigma_{L-2}^Y \sigma_{L-1}^Y \sigma_0^Y + \sigma_{L-1}^Y \sigma_0^Y \sigma_1^Y \Bigg] \Bigg\}
\end{aligned} \tag{2.3}
$$

where α is the constant of anisotropy and we have included explicitly the boundary terms. In (2.3) and hereafter we are assuming L as being a multiple of 6 in order to ensure the symmetries of the model.

The spectra of the above Hamiltonian for low and high values of λ are related by

$$H_{XY}^{(3)}(\alpha, \lambda) = \lambda H_{XY}^{(3)}(\alpha, 1/\lambda) \tag{2.4}$$

This property is derived by making, in (2.3), the $SU(2)$ spin rotation $\{\sigma_i^X \to \sigma_i^Y,\ \sigma_i^Y \to \sigma_i^X,\ \sigma_i^Z \to -\sigma_i^Z;\ i = 1, 2,..., L)$. It is interesting to point out here that (2.4) is the analogous relation between the low and high temperatures of self-dual models like the Ising and Potts models. However, (2.4) is exact for all the eigenstates of (2.3) with L finite, and not only for part of them as usual for the self-dual models.

The standard XY model has a $U(1)$ symmetry because the Hamiltonian (2.2) commutes with the z component of the total spin operator $S_Z = \sum \sigma_i^Z$. The triplet model (2.3), on the other hand, can be expressed in terms of two other Hamiltonians, each one having separately half of the three-spin interactions and invariance under a local gauge symmetry. In order to see this gauge, let us define the following new variables:

$$
\begin{aligned}
\eta_i^a &= \sigma_{2i}^a \sigma_{2i+1}^a \sigma_{2i+2}^a; \qquad i = 0, 1,..., L/2-2 \\
\eta_{L/2-1}^a &= \sigma_{L-2}^a \sigma_{L-1}^a \sigma_0^a \\
\xi_i^a &= \sigma_{2i+3}^a \sigma_{2i+4}^a \sigma_{2i+5}^a; \qquad i = 0, 1,..., L/2-3 \\
\xi_{L/2-2}^a &= \sigma_{L-1}^a \sigma_0^a \sigma_1^a \\
\xi_{L/2-1}^a &= \sigma_1^a \sigma_2^a \sigma_3^a
\end{aligned} \tag{2.5}
$$

where $a=X, Y$. Is it clear from their definition that the variables η and ξ commute, i.e.,

$$[\eta_i^X, \xi_j^Y] = [\eta_i^Y, \xi_j^X] = [\eta_i^X, \xi_j^X] = [\eta_i^Y, \xi_j^Y] = 0 \tag{2.6a}$$

$$[\xi_i^X, \xi_j^X] = [\xi_i^Y, \xi_j^Y] = [\eta_i^X, \eta_j^X] = [\eta_i^Y, \eta_j^Y] = 0 \tag{2.6b}$$

for all pairs $(i, j) \in \{1, 2,..., L\}$, and

$$[\xi_i^X, \xi_j^Y] = [\eta_i^X, \eta_j^Y] = 0 \tag{2.6c}$$

unless $i=j$, $|i-j|=1$, or $(i, j)=(0, L/2-1)$ or $(L/2-1, 0)$, in which cases

$$\{\xi_i^X, \xi_j^Y\} = \{\eta_i^X, \eta_j^Y\} = 0 \tag{2.6d}$$

In terms of these new variables the Hamiltonian (2.3) takes the simple form

$$H_{XY}^{(3)} = H_{XY}^{(A)}(\eta) + H_{XY}^{(A)}(\xi) \tag{2.7a}$$

where

$$H_{XY}^{(A)} = -\sum_{i=0}^{L/6-1} [(\alpha\eta_{3i}^X + \eta_{3i+1}^X + \alpha\eta_{3i+2}^X) + \lambda(\alpha\eta_{3i}^Y + \eta_{3i+1}^Y + \alpha\eta_{3i+2}^Y)] \tag{2.7b}$$

$$H_{XY}^{(A)} = -\sum_{i=0}^{L/6-1} [(\alpha\xi_{3i}^X + \xi_{3i+1}^X + \alpha\xi_{3i+2}^X) + \lambda(\alpha\xi_{3i}^Y + \xi_{3i+1}^Y + \alpha\xi_{3i+2}^Y)] \tag{2.7c}$$

However, due to the periodic boundary condition, although the variables ξ and η commute, they are not independent, because they should satisfy the constraints

$$\prod_{i=0}^{L/6-1} \eta_{3i}^a \eta_{3i+2}^a \xi_{3i}^a \xi_{3i+2}^a = 1; \qquad a=X, Y \tag{2.8a}$$

$$\prod_{i=0}^{L/6-1} \eta_{3i}^a \eta_{3i+1}^a \xi_{3i}^a \xi_{3i+1}^a = 1; \qquad a=X, Y \tag{2.8b}$$

From (2.6)–(2.8) the triplet model (2.3) can therefore be expressed in terms of two decoupled Hamiltonians satisfying the constraints (2.8). In the next section we study these decoupled Hamiltonians.

3. THE ALTERNATED TRIPLET *XY* MODEL

In the last section we showed that the triplet XY model can be described in terms of two decoupled Hamiltonians, $H_{XY}^{(A)}(\eta)$ and $H_{XY}^{(A)}(\xi)$, subject to the constraints (2.8), each of these Hamiltonians having half of the triplet interactions of (2.3). These Hamiltonians are alternate triplet XY models, because in terms of the original variables σ^X and σ^Y, they can be expressed as

$$H_{XY}^{(A)}(\eta) = -\sum_{i=0}^{L/2-1} (\sigma_{2i}^X \sigma_{2i+1}^X \sigma_{2i+2}^X + \lambda \sigma_{2i}^Y \sigma_{2i+1}^Y \sigma_{2i+2}^Y) \tag{3.1a}$$

and

$$H_{XY}^{(A)}(\xi) = -\sum_{i=0}^{L/2-1} (\sigma_{2i+1}^X \sigma_{2i+2}^X \sigma_{2i+3}^X + \lambda \sigma_{2i+1}^Y \sigma_{2i+2}^Y \sigma_{2i+3}^Y) \tag{3.1b}$$

with periodic boundary conditions imposed. Consequently, in order to study the triplet XY model, we need first to analyze the alternate triplet XY models (3.1) and second couple them according to the constraints (2.8).

Let us consider the alternate model (3.1a). This Hamiltonian has a $Z(2)$ local gauge invariance. The local gauge operators, in terms of the variables σ^X and σ^Y, are given by

$$\begin{aligned} G_i^a &= \sigma_{2i-1}^a \sigma_{2i}^a \sigma_{2i+1}^a; \qquad & i &= 1, 2, \ldots, L/2-1 \\ G_0^a &= \sigma_{L-1}^a \sigma_0^a \sigma_1^a; \qquad & a &= X, Y \end{aligned} \tag{3.2}$$

In Fig. 1 we illustrate the location of these gauge operators with respect to the interactions (wave lines). The commutation of these gauge operators with the Hamiltonian (3.1a) arises because, as we can see from Fig. 1, they have in common with the interactions an even number (0 or 2) of Pauli matrices. This gauge invariance implies that, by choosing the basis in which σ^Y is diagonal, we can separate the Hilbert space associated with (3.1a) in $2^{L/2}$ block-disjoint sectors labeled by the eigenvalues ($+1$ or -1) of the $Z(2)$ local operators G_i ($i = 0, 1, \ldots, L/2-1$). In the next section we relate the alternate triplet model with the anisotropic triplet Ising model.[3]

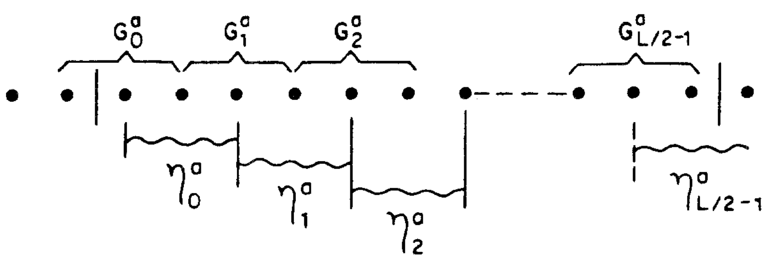

Fig. 1. Location of interactions in the Hamiltonian (3.1a) (wavy lines) and the gauge operators (3.2).

4. THE TRIPLET ISING MODEL

We show in this section that the alternate triplet XY models (3.1) are related to the triplet Ising model,[3] whose quantum Hamiltonian is defined by

$$\begin{aligned} H_{\text{ISING}}^{(3)} = &- \sum_{i=0}^{L/3-6} (\sigma_{3i}^Z \sigma_{3i+1}^Z \sigma_{3i+2}^Z + \alpha\sigma_{3i+1}^Z \sigma_{3i+2}^Z \sigma_{3i+3}^Z + \alpha\sigma_{3i+2}^Z \sigma_{3i+3}^Z \sigma_{3i+4}^Z) \\ &+ \sigma_{L-3}^Z \sigma_{L-2}^Z \sigma_{L-1}^Z + \alpha C_1 \sigma_{L-2}^Z {}_{L-1}^{Z} \sigma_0^Z + \alpha C_2 \sigma_{L-1}^Z \sigma_0^Z \sigma_1^Z \\ &- \lambda \sum_{i=0}^{L/3-6} (\alpha\sigma_{3i}^X + \sigma_{3i+1}^X + \alpha\sigma_{3i+2}^X) \end{aligned} \tag{4.1}$$

where, as before, λ plays the role of the temperature and C_1, C_2, which may assume the values $+1$, -1, or 0, specify the boundary conditions. In the periodic case $C_1 = C_2 = 1$, while in the case of free boundary conditions $C_1 = C_2 = 0$. This Hamiltonian has a $Z(2) \otimes Z(2)$ nonlocal symmetry due to the commutation with the $Z(2)$ operators

$$\prod_{i=0}^{L/3-1} \sigma_{3i}^X \sigma_{3i+1}^X = \hat{P}_1 \tag{4.2a}$$

$$\prod_{i=0}^{L/3-1} \sigma_{3i}^X \sigma_{3i+2}^X = \hat{P}_2 \tag{4.2b}$$

$$\prod_{i=0}^{L/3-1} \sigma_{3i+1}^X \sigma_{3i+2}^X = \hat{P}_1 \cdot \hat{P}_2 = \hat{P}_3 \tag{4.2c}$$

Its Hilbert space can therefore be separated into four disjoint sectors according to the eigenvalues (± 1) of $\hat{P}_1$ and $\hat{P}_2$. The finite-size studies of this model reveal[3] that the phase diagram and critical exponents of (4.1) are similar to those of the Ashkin–Teller model. For $0 < \alpha \leqslant 1$ the line $\lambda = \lambda_c = 1$ is a critical line where the exponents vary continuously, while for $\alpha > 1$ the model shows two phase transitions at $\lambda = \lambda_c^{(1)}(\alpha)$ and $\lambda = \lambda_c^{(2)}(\alpha)$ with an intermediate phase where the $Z(2) \otimes Z(2)$ symmetry is partially brocken. These two phase transitions have the critical exponents of the standard (two-spin interactions) Ising model.

In order to continue our analysis, we introduce the new variables

$$\eta_{i+1}^Y = \sigma_i^Z \sigma_{i+1}^Z \sigma_{i+2}^Z; \qquad i = 0, 1,..., L-3 \tag{4.3a}$$

$$\eta_{L-1}^Y = C_1 \sigma_{L-2}^Z \sigma_{L-1}^Z \sigma_0^Z \tag{4.3b}$$

$$\eta_0^Y = C_2 \sigma_{L-1}^Z \sigma_0^Z \sigma_1^Z \tag{4.3c}$$

$$\eta_i^X = \sigma_i^X; \qquad i = 0, 1,..., L-1 \tag{4.3d}$$

which obey the same algebra (2.6) as the variables η defined in (2.5). In terms of these variables the Hamiltonian (4.1) takes the form

$$H_{\mathrm{ISING}}^{(3)} = -\sum_{i=0}^{L/3-1} [(\alpha\eta_{3i}^X + \eta_{3i+1}^X + \alpha\eta_{3i+2}^X) + \lambda(\alpha\eta_{3i}^Y + \eta_{3i+1}^Y + \alpha\eta_{3i+2}^Y)] \tag{4.4}$$

and the parity operators (4.2) are given by

$$\prod_{i=0}^{L/3-1} \eta_{3i}^X \eta_{3i+1}^X = \hat{P}_1 \tag{4.5a}$$

$$\prod_{i=0}^{L/3-1} \eta_{3i}^X \eta_{3i+2}^X = \hat{P}_2 \tag{4.5b}$$

In the case of toroidal boundary conditions (C_1, $C_2 \neq 0$) the variables η_i^Y ($i = 0, 1,..., L-1$) are not independent, but should satisfy the constraints

$$\prod_{i=0}^{L/3-1} \eta_{3i+1}^Y \eta_{3i+2}^Y = C_1 \tag{4.6a}$$

$$\prod_{i=0}^{L/3-1} \eta_{3i}^Y \eta_{3i+2}^Y = C_2 \tag{4.6b}$$

Equations (2.7b) and (4.4) show that the alternate triplet XY Hamiltonian (see Section 3) in a $2L$-site chain and the triplet Ising Hamiltonian in an L-site chain are the same when expressed in terms of the variables η. However, while these variables are independent in the first model, in the triplet Ising model they should satisfy the constraints (4.6). In order to see these relations more clearly, let us rename, in the alternated XY model (3.1), the original variables σ^X and σ^Y. The variables σ_{2i}^a ($i = 0, 1,..., L/2-1$, $a = X, Y$), in the even sites, we rename ν_i^a, and the variables σ_{2i+1}^a ($i = 0, 1,..., L/2-1$; $a = X, Y$), in the odd sites, we rename μ_i^a:

$$\eta_i^a = \nu_{i-1}^a \mu_i^a \nu_{i+1}^a; \qquad a = X, Y; \quad i = 1, 2,..., L/2-1 \tag{4.7}$$

The gauge operators (3.2) and the $\{\eta^Y\}$ variables (2.5) are now given by

$$G_i^Y = \mu_{i-1}^Y \nu_i^Y \mu_i^Y; \qquad i = 1, 2,..., L/2-1; \qquad G_0^Y = \mu_{L/2-1}^Y \nu_0^Y \mu_0^Y \tag{4.8a}$$

$$\eta_i^Y = G_i^Y G_{i+1}^Y \mu_{i-1}^Y \mu_i^Y \mu_{i+1}^Y; \qquad \eta_0^Y = G_0^Y G_1^Y \mu_{L/2-2}^Y \mu_0^Y \mu_1^Y \tag{4.8b}$$

and

$$\eta_{L/2-1}^Y = G_{L/2-1}^Y G_0^Y \mu_{L/2-2}^Y \mu_{L/2-1}^Y \mu_0^Y \tag{4.8c}$$

These equations show that if we choose the basis in which σ^Y (or μ^Y and ν^Y) are diagonal, the gauge operators G_i^Y are fixed (± 1) and the diagonal part of (2.7b) or (3.1a) will involve only the μ_i^Y operators. Because of this, the effect of the nondiagonal variables η^X is the same as the effect of μ^X, and we can write the triplet alternate XY Hamiltonian as

$$\begin{aligned}
H_{XY}^{(A)} = &-\sum_{i=0}^{L/6-1} (\alpha\mu_{3i}^X + \mu_{3i+1}^X + \alpha\mu_{3i+2}^X) \\
&-\lambda\left[\sum_{i=0}^{L/6-2} (G_{3i+1}^Y G_{3i+2}^Y \mu_{3i}^Y \mu_{3i+1}^Y \mu_{3i+2}^Y \right. \\
&+\alpha G_{3i+2}^Y G_{3i+3}^Y \mu_{3i+1}^Y \mu_{3i+2}^Y \mu_{3i+3}^Y \\
&+\alpha G_{3i+3}^Y G_{3i+4}^Y \mu_{3i+2}^Y \mu_{3i+3}^Y \mu_{3i+4}^Y) \\
&+G_{L/2-2}^Y G_{L/2-1}^Y \mu_{L/2-3}^Y \mu_{L/2-2}^Y \mu_{L/2-1}^Y \\
&\left.+\alpha G_{L/2-1}^Y G_0^Y \mu_{L/2-2}^Y \mu_{L/2-1}^Y \mu_0^Y + G_0^Y G_1^Y \mu_{L/2-1}^Y \mu_0^Y \mu_1^Y\right]
\end{aligned} \tag{4.9}$$

It is interesting to observe that even after we fix the values of the gauge sectors the operators

$$\hat{P}_1 = \prod_{i=0}^{L/3-1} \eta_{3i}^X \eta_{3i+1}^X = \prod_{i=0}^{L/3-1} \mu_{3i}^X \mu_{3i+1}^X \tag{4.10a}$$

$$\hat{P}_2 = \prod_{i=0}^{L/3-1} \eta_{3i}^X \eta_{3i+2}^X = \prod_{i=0}^{L/3-1} \mu_{3i}^X \mu_{3i+1}^X \tag{4.10b}$$

still commute with the Hamiltonian (4.9). These operators are the same parity operators introduced above in (4.5) for the triplet Ising model. The constraint equations (4.6) for the triplet Ising model correspond in the alternated triplet model to the conditions

$$\prod_{i=0}^{L/3-1} \eta_{3i+1}^Y \eta_{3i+2}^Y = \prod_{i=0}^{L/3-1} G_{3i}^Y G_{3i+1}^Y \tag{4.11a}$$

$$\prod_{i=0}^{L/3-1} \eta_{3i}^Y \eta_{3i+1}^Y = \prod_{i=0}^{L/3-1} G_{3i}^Y G_{3i+1}^Y \tag{4.11b}$$

These results show that the $2^{L/2}$ sectors of the triplet model characterized by the gauge values $\{G_i^Y\}$ will be separated into four distinct groups of $2^{L/2}/4$ sectors characterized by the product (4.11) of the gauges. From (2.7), (4.4), and (4.6) the sectors in each of these groups are degenerate and equivalent to the triplet Ising model with toroidal boundary conditions C_1

and C_2 given by the values of the products of the gauges (4.11). As in the triplet Ising model, each of these distinct sectors is still block separated into four other disjoint sectors labeled by the eigenvalues of the parity operators $\hat{P}_1$ and $\hat{P}_2$ given in (4.10). Consequently, in order to study the eigenspectra of the alternate triplet XY chain with periodic boundaries and $2L$ sites we should study equivalently the triplet Ising Hamiltonian (4.1) with L sites and general toroidal boundary conditions, which will be done in the next section.

Before closing this section, let us discuss some equivalences of sectors for the triplet Ising Hamiltonian. The canonical transformations $\eta_i^X \to \eta_i^Y$ and $\eta_i^Y \to \eta_i^X$ $(i=0, 1,..., L-1)$ give us a relation analogous to (2.4). However, Eqs. (4.5) and (4.6), which characterizes the sectors and boundary conditions, are changed. At the special point $(\lambda = 1)$ we have

$$H_{P_1, P_2}^{C_1, C_2} = H_{C_2, C_1 C_2}^{P_1 P_2, P_1} \tag{4.12}$$

which says that the sector where the parity operators (4.2a)–(4.2b) have the eigenvalues P_1 and P_2 (± 1) of the triplet Ising Hamiltonian with boundary conditions C_1 and C_2 (± 1) has the same eigenspectrum as the sector where the parity operators have the eigenvalues C_2 and $C_1 C_2$ and the boundary conditions are $P_1 P_2$ and P_2. Another interesting relation among sectors can be obtained by making a reflection in the lattice: $i \to (L-1) - i$ $(i=0, 1,..., L-1)$. With this transformation we obtain

$$H_{P_1, P_2}^{C_1, C_2} = H_{P_1 P_2, P_2}^{C_2, C_1} \tag{4.13}$$

where the notation is the same as in (4.12). For completeness, in the case of free boundaries, the triplet Ising model has the property

$$H_{P_1, P_2}^{(F)} = H_{P_1 P_2, P_2}^{(F)} \tag{4.14}$$

which states that the sectors with parities P_1 and P_2 are equivalent to those with parities $P_1 P_2$ and P_2.

5. CONFORMAL INVARIANCE AND MASS GAP AMPLITUDES

The assumption that most of statistical mechanics systems at criticality are conformally invariant[5-7] produced remarkable results in two dimensions. Specifically, Cardy[8,9] has derived a set of important relations between the finite-size corrections of the transfer matrix in a strip of finite width and the conformal anomaly and scaling dimensions of the operators describing the critical behavior of the infinite system.

The pertinent results for our purposes, when transcribed to the quantum Hamiltonian formalism,[10] are as follows. Corresponding to each primary operator $\phi_{\Delta,\bar{\Delta}}$ in the conformal algebra of the infinite system with dimension $X_\phi = \Delta + \Delta$ and spin $S_\phi = \Delta - \bar{\Delta}$, there exists at the critical point ($\lambda = \lambda_c$) a set of eigenstates of the L-site quantum Hamiltonian chain with toroidal boundary conditions with eigenenergies $E_{n,\bar{n}}$ and momenta $P_{n,\bar{n}}$ given by

$$\begin{aligned} E_{n,\bar{n}} &= E_0 + \zeta \frac{2\pi}{L}(X_\phi + n + \bar{n}) + o(L^{-1}) \\ P_{n,\bar{n}} &= \frac{2\pi}{L}(S_\phi + n - \bar{n}); \qquad n, \bar{n} = 0, 1, 2,... \end{aligned} \tag{5.1}$$

where E_0 is the ground-state energy for the finite chain with periodic boundaries and ζ is a model-dependent constant, reflecting the fact that the singular behavior of the Hamiltonian is insensitive to multiplication by an arbitrary constant. Also, as a consequence of the conformal invariance of the infinite system the finite-size corrections of the ground-state energy is proportional to the conformal anomaly, or central charge of the conformal theory governing the criticality of the statistical model. For periodic boundaries the ground-state energy $E_0(L)$ at the critical point behaves assymptotically like

$$E_0/L = e_\infty - \zeta\pi c/6L^2 + o(L^{-2}) \tag{5.2}$$

where e_∞ is the energy per site in the bulk limit.

In the following we use the relations (5.1) and (5.2) to obtain the scaling dimensions for the triplet Ising model, the alternate triplet XY model, and the triplet XY model. Because the eigenspectra of the last two models are related with that of the triplet Ising model, this model will be considered initially.

5.1. The Triplet Ising Model

From the finite-size calculations[3] for the triplet Ising model with periodic boundaries, we know that this model exhibits two types of critical behavior. For $0 < \alpha \leqslant 1$ the model has a critical line at $\lambda = \lambda_c = 1$ governed by a $c = 1$ conformal theory, having continuously varying critical exponents. For $\alpha > 1$ the model undergoes two phase transitions, at $\lambda = \lambda_c^{(1)}(\alpha)$ and $\lambda = \lambda_c^{(2)}(\alpha)$, with the critical exponents of the standard Ising model ($c = 1/2$). As we saw in the last section, in order to study the triplet XY model we need to calculate the dimensions arising in the triplet Ising chain with general boundary conditions imposed.

For the purpose of using (5.1) we have to introduce, for general toroidal boundary conditions, eigenstates with momentum quantum numbers. If we choose in (4.1) the σ^X basis, the basis vectors with momenta $P = p + \Lambda$ ($p = 0, 1,..., L/3 - 1$) in units of $2\pi/(L/3)$ are given by

$$\begin{aligned} |p, \{s\}\rangle = \frac{1}{(L/3)^{1/2}} \{ & |s_{0,0}s_{0,1}s_{0,2}; s_{1,0}s_{1,1}s_{1,2}; ...; s_{L/3-1,0}s_{L/3-1,1}s_{L/3-1,2}\rangle \\ & + e^{i[2\pi Y_1/(L/3)]} |s_{1,0}s_{1,1}s_{1,2}; ...; s_{L/3-1,0}s_{L/3-1,1}s_{L/3-1,2}; \\ & \times s_{0,0}s_{0,1}s_{0,2}\rangle + \cdots \\ & + e^{i[2\pi Y_{L/3-1}/(L/3)]} |s_{L/3-1,0}s_{L/3-1,1}s_{L/3-1,2}; s_{0,0}s_{0,1}s_{0,2}; ...; \\ & \times s_{L/3-2,0}s_{L/3-2,1}s_{L/3-2,2}\rangle \} \end{aligned} \tag{5.3}$$

where $s_{n,k}$ is the eigenvalue (± 1) of the operator σ_i^X in the site $i = 3n + k$ ($i = 0, 1, 2,..., L - 1$) and the constants Y_i ($i = 1, 2,..., L/3 - 1$) are given by

$$Y_i = \left(\frac{\tilde{Q}_0}{2} \sum_{k=0}^{i-1} s_{k,0} + \frac{\tilde{Q}_1}{2} \sum_{k=0}^{i-1} s_{k,1} + \frac{\tilde{Q}_2}{2} \sum_{k=0}^{i-1} s_{k,2} \right) - (\Lambda - p)i \tag{5.4}$$

The constants $\tilde{Q}_i$, $i = 0, 1, 2$, and Λ will depend upon the particular boundary condition and sector, and in Table I we show their values for the possible toroidal boundary conditions.

From relations (5.1) the several anomalous dimensions of the operators can be obtained by extrapolating the sequences

$$G_{n,P_1P_2}^{C_1C_2}(L, p) = E_{n,P_1P_2}^{C_1C_2}(L, p) - E_{1,++}^{++}(L, 0) = \frac{2\pi\zeta_1}{L} X + o(L^{-1}) \tag{5.5}$$

where we denote by $E_{n,P_1P_2}^{C_1C_2}(L, p)$ the nth eigenenergy ($n = 1, 2,...$) with momentum $P = [2\pi/(L/3)](p + \Lambda)$, ($p = 0, 1,..., L/3 - 1$) (see Table I) of the sector with parities P_1 and P_2 ($\pm$) of the Hamiltonian with boundary conditions C_1 and C_2 ($\pm$). In (5.5) the ground state for a periodic boundary is $E_{1,++}^{++}(L, 0)$ and X is the corresponding scaling dimension. In

Table I. Value of Constants in (5.4) for Several Boundary Conditions

C_1	C_2	$\tilde{Q}_0$	$\tilde{Q}_1$	$\tilde{Q}_2$	$\Lambda(P_i)$
+1	+1	0	0	0	0
−1	+1	1	1	0	$(1 - P_1)/4$
−1	−1	1	0	1	$(1 - P_2)/4$
+1	−1	0	1	1	$(1 - P_3)/4$

order to extract the anomalous dimensions from (5.5), we need to calculate the constant ζ_{I}. This can be done [see (5.1)] by extrapolating the difference between higher energy states associated with the same primary operator[11]; for example, we can use the sequence

$$Z_L = E_{1,-+}^{++}(L,1) - E_{1,-+}^{++}(L,0) = \frac{2\pi}{L}\zeta_{\mathrm{I}} + o(L^{-1}) \tag{5.6}$$

which is the mass gap amplitude between the two lowest states in the sector $P_1 = +1$, $P_2 = -1$ for the Hamiltonian with periodic boundary condition $C_1 = C_2 = +1$.

The case of periodic boundaries was already studied.[3] From the sequences $G_{2,-+}^{++}(L,0)$, $G_{1,-+}^{++}(L,0)$, and $G_{1,--}^{++}(L,0)$ the dimensions X_ε, $X_\bigcirc$, and $X_\square$ were conjectured

$$X_\varepsilon = \frac{\pi}{2\cos^{-1}(-\alpha)} \tag{5.7a}$$

$$X_\bigcirc = \frac{1}{8}, \qquad X_\square = \frac{X_\varepsilon}{4}, \qquad 0 < \alpha \leqslant 1 \tag{5.7b}$$

while for $\alpha > 1$ we have two phase transitions of Ising type where $X_\varepsilon = 1$ and $X_\bigcirc = 1/8$ or $X_\square = 1/8$ in each of the phase transitions. The dimension $X_\varepsilon = 2 - 1/\nu$ is associated to the energy operator, where ν is the correlation-length exponent, while the dimensions $X_\bigcirc$ and $X_\square$ govern the spin–spin correlations in different sublattices.[3] The fact that for $0 < \alpha \leqslant 1$ the exponents depend continuously on the anisotropy induces us to expect, in terms of standard renormalization group arguments,[12] the existence of a marginal operator ($X_{\mathrm{mar}} = 2$) governing the motion along the fixed line. In fact, in Table II we show, for some values of α, the sequence

Table II. Finite-Size Sequences $G_{3,++}^{++}(L,0)/Z_L$ for $L = 6$–18 and Some Values of α[a]

L	$\alpha = 0.1$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 1.0$
6	2.822095	2.668088	2.408439	3.092158
9	2.306231	2.237341	2.111631	2.886157
12	2.162053	2.111954	2.027738	2.594960
15	2.100935	2.065327	2.002612	2.551423
18	2.069084	2.042669	1.993185	2.523459
Estimate	2.00	1.96	1.99	2.4
Conjecture	2	2	2	2

[a] The conjectured value is $X_{\mathrm{mar}} = 2$.

$G^{++}_{3,++}(L,0)/Z_L$, indicating the presence of such a marginal operator. All the eigenspectra calculations reported in this paper were done by using Lanczo's method and the extrapolations by using the van den Broeck–Schwartz approximants[13,14] (for a recent review of these methods see ref. 15). The slow convergence in this table for α around unity is expected[3] because at $\alpha=1$ the operator responsible for the corrections to scaling, usually irrelevant, becomes marginal, giving rise to logarithmic corrections.[16]

The mass gap amplitudes, in the case of nonperiodic, but toroidal boundary conditions will be related to the parafermions.[7] From the relations (4.13) and (4.14) we have only studied the dimensions associated with the low-lying states of sectors and boundary conditions that are not related to the periodic case. The most interesting of these sequences are

$$G^{--}_{1,-+}(L,1)/Z_L \to X^{S=1}_{\mathrm{Pf}} \tag{5.8a}$$

$$G^{-+}_{1,-+}(L,0)/Z_L \to X^{S=1/2}_{\mathrm{Pf}} \tag{5.8b}$$

$$G^{-+}_{1,--}(L,0)Z_L \to \tilde{X}^{S=1/2}_{\mathrm{Pf}} \tag{5.8c}$$

In Tables III–V we show their finite values together with the extrapolated and conjectured results. The lowest energy in the sector $P_1=-1$ and $P_2=+1$, with boundary condition $C_1=C_2=-1$, is a state with momentum $P=2\pi/(L/3)$, which corresponds, in (5.4), to $p=1$ and $\Lambda=0$ (see Table I). The mass gap amplitude associated with this state, according to (5.1), is related to a spin-1 operator. The extrapolation in Table III indicates the conjecture

$$X^{S=1}_{\mathrm{Pf}}=1, \qquad 0<\alpha\leqslant 1 \tag{5.9a}$$

Table III. Finite-Size Sequences $G^{--}_{1,-+}(L,1)/Z_L$ for $L=6$–18 and Some Values of α[a]

L	$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=1.0$
6	1.362190	1.208766	1.131648	1.067865
9	1.150097	1.102541	1.067132	1.040306
12	1.079906	1.054037	1.034169	1.024972
15	1.049868	1.033081	1.019664	1.017238
18	1.034163	1.022230	1.012180	1.012831
Estimate	0.985	1.001	1.000	1.001
Conjecture	1	1	1	1

[a] The conjectured value is $X^{S=1}_{\mathrm{Pf}} \doteq 1$.

Table IV. Finite-Size Sequences $G^{-+}_{1,++}(L,0)/Z_L$ for $L=6$–18 and Some Values of α[a]

L	$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$
6	0.910617	0.867922	0.825400	0.784921
9	0.730000	0.705514	0.680071	0.654032
12	0.680576	0.664966	0.647076	0.626552
15	0.659607	0.648688	0.634718	0.616851
18	0.648682	0.640522	0.628929	0.612642
Estimate	0.627	0.625	0.624	0.610
Conjecture	0.625	0.625	0.625	0.625

[a] The conjectured value is $X_{\mathrm{Pf}}^{S=1/2}=5/8$.

On the other hand, the lowest-energy states in the sector $P_1=+1$, $P_2=-1$ ($P_1=P_2=-1$) with boundary condition specified by $C_1=-1$, $C_2=+1$ are states with momenta $P=\frac{1}{2}\cdot 2\pi/(L/3)$ because $p=0$ and $\Lambda=1/2$ (see Table I). Consequently, from (5.1), they should correspond to spin $S=1/2$ parafermion operators. The extrapolated results of Tables IV and V indicate the conjecture

$$X_{\mathrm{Pf}}^{S=1/2}=\frac{5}{8}=0.625, \qquad 0<\alpha\leqslant 1 \tag{5.9b}$$

and

$$\tilde{X}_{\mathrm{Pf}}^{S=1/2}=X_{\square}+\frac{1}{16X_{\square}}, \qquad 0<\alpha\leqslant 1 \tag{5.9c}$$

where $X_{\square}$ is given by (5.7c).

Table V. Finite-Size Sequences $G^{-+}_{1,--}(L,0)/Z_L$ for $L=6$–18 and Some Values of α[a]

L	$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$
6	0.706453	0.704648	0.710382	0.726505
9	0.577100	0.579256	0.587834	0.605234
12	0.541377	0.547631	0.559111	0.578593
15	0.526172	0.534820	0.547880	0.568466
18	0.518237	0.528334	0.542333	0.563578
Estimate	0.502	0.515	0.532	0.555
Conjecture	0.500	0.513	0.529	0.554

[a] The conjectured value is $\tilde{X}_{\mathrm{Pf}}^{S=1/2}=X_{\square}+1/(16X_{\square})$, where $X_{\square}=\pi/[8\cos^{-1}(-\alpha)]$.

The dimensions $X_{\rm Pf}^{S=1}$ and $X_{\rm Pf}^{S=1/2}$ have the same value as the dimensions of the parafermions of spin 1 and 1/2 in the Ashkin–Teller model.[11] It is also known[17] that beyond the spin-1 and spin-1/2 parafermionic operators, with dimension given by (5.9a), (5.9b) the Ashkin–Teller model also exhibits parafermions with spins $S=1/4$ and $S=3/4$. If the similarities between the triplet Ising model and the Ashkin–Teller model were complete, we should find the corresponding dimensions[17] $X_{\rm Pf}^{S=1/4}= X_{\Box}/4+1/16X_{\Box}$ and $X_{\rm Pf}^{S=3/4}=9X_{\Box}/4+1/16X_{\Box}$. However, these dimensions do not occur, and we can understand this fact from the momentum states in the triplet Ising model with general toroidal boundary conditions. From (5.1), in order to obtain parafermions of spin 1/4 or 3/4 we should have states with momentum

$$P=(p+\Lambda)\frac{2\pi}{L/3}=\frac{2\pi}{L/3}\frac{1}{4}$$

or

$$P=\frac{2\pi}{L/3}\frac{3}{4}$$

which is not possible because, as we can see in Table I, Λ is integer or half-integer.

For completeness we have also studied the triplet Ising model with free ends. In this case the mass gap amplitudes are related to the surface exponents X_S. To each surface exponent of the infinite system[7,18] there corresponds a set of states in the free boundary Hamiltonian on L sites with energies at the bulk critical point given by

$$E_r^{(F)}(L)=E_0^{(F)}(L)+\pi\zeta_{\rm I}(X_S+r)/L+o(L^{-1}); \qquad r=0,1,2,... \tag{5.10}$$

Here $E_0^{(F)}(L)$ is the ground-state energy of the finite chain and $\zeta_{\rm I}$ is the same constant appearing in (5.1) and (5.2), which can be estimated by extrapolating the sequence

$$W_L=E_{1,-+}^{(F)}-E_{0,-+}^{(F)}=\pi\zeta_{\rm I}/L+o(L^{-1}) \tag{5.11}$$

where $E_{0,++}^{(F)}$ $(E_{1,++}^{(F)})$ is the ground-state energy (first excited state) in the sector $P_1=P_2=+1$ of the Hamiltonian with free ends. The surface exponents corresponding to the low-lying states in each sector can be obtained from the extrapolated value of the following sequences

$$S_L^0=E_{1,++}^{(F)}-E_{0,++}^{(F)}=\frac{\pi}{L}\zeta_{\rm I}X_\varepsilon^{(S)}+o(L^{-1}) \tag{5.12a}$$

$$S_L^1 = E_{0,--}^{(F)} - E_{0,++}^{(F)} = \frac{\pi}{L}\zeta_1 X_{\square}^{(S)} + o(L^{-1}) \tag{5.12b}$$

$$S_L^2 = E_{0,-+}^{(F)} - E_{0,++}^{(F)} = \frac{\pi}{L}\zeta_1 X_{\bigcirc}^{(S)} + o(L^{-1}) \tag{5.12c}$$

where $E_{n,P_1P_2}^{(F)}$ is the nth excited state in the sector P_1, P_2 of the Hamiltonian with free boundary condition. In Table VI we present, for some values of α, the estimates obtained from the extrapolation of the sequences S_L^0/W_L, S_L^1/W_L, and S_L^2/W_L, using lattices sizes $L = 6$–15. These results indicate the conjecture

$$X_\varepsilon^{(S)} = 2 \tag{5.13a}$$

$$X_{\square}^{(S)} = \cos^{-1}(-\alpha)/\pi = 1/(2X_\varepsilon) \tag{5.13b}$$

$$X_{\bigcirc}^{(S)} = 1 \tag{5.13c}$$

for $0 < \alpha \leqslant 1$. The value $X_\varepsilon^{(S)} = 2$ is expected for all $(1+1)$-dimensional critical models[7] and the exponents $X_{\square}^{(S)}$ and $X_{\bigcirc}^{(S)}$ have exactly the same value as the surface exponents which correspond to the magnetization and polarization correlations in the Ashkin–Teller model.[19,20] The equivalence between the two models with respect to the surface exponents seems to be complete.

5.2. The Alternate Triplet *XY* Model

From the results of Section 4, the Hilbert space associated with the alternated XY model with L sites and periodic boundaries is block

Table VI. Finite-Size Estimates of S_L^0/W_L, S_L^1/W_L, and S_L^2/W_L for $L = 6$–15 and Some Values of α[a]

	$X_\varepsilon^{(S)}$		$X_{\square}^{(S)}$		$X_{\bigcirc}^{(S)}$	
α	Estimate	Conjecture	Estimate	Conjecture	Estimate	Conjecture
0.1	1.999	2	0.525	0.53188...	0.999	1
0.2	1.998	2	0.564	0.56409...	1.002	1
0.4	1.998	2	0.625	0.63098...	1.003	1
0.5	2.002	2	0.665	0.66666...	1.001	1
0.6	2.002	2	0.699	0.70483...	0.998	1
0.8	2.006	2	0.772	0.79516...	0.95	1

[a] The conjectured values are given by (5.13).

separated in $2^{L/2}$ sectors of dimension $2^{L/2}$ labeled by the gauge choice $\{G_0^Y, G_1^Y,..., G_{L/2-1}^Y\}$. These sectors are divided in four groups of $2^{L/2}/4$ degenerate sectors, each of these groups having a different value for the product of the gauge variables in even and odd sites of the lattice [see (4.11)]. From (4.10) and (4.11) these distinct sets of sectors have the same spectra as the triplet Ising model with $L/2$ sites and different toroidal boundary conditions. Consequently, the phase diagrams of both models are the same in the plane λ, α.

From the above equivalences, Eqs. (5.1) and (5.11), we conclude that the criticality of the alternate triplet XY model is governed by the same conformal theory as the triplet Ising model. The operator content of the alternate triplet XY model with periodic boundaries is the sum of the operator contents of the triplet Ising chain with arbitrary toroidal boundary conditions. In particular, all the dimensions obtained in (5.7), (5.9), and (5.11) are also present in the alternate triplet XY model.

5.3. The Triplet *XY* Model

From the results of Section 2, the triplet XY model can be written as the direct sum of two alternate triplet XY models subject to the constraints (2.8). These constraints, from (4.10) and (4.11), imply that in order to obtain the spectrum of the triplet XY model, we should combine identical sectors of the two alternate models. Equivalently, we should combine the spectra of two triplet Ising chains with $L/2$ sites having the same boundary conditions C_1, C_2 and parities P_1, P_2. Using the notation of Eq. (5.5), we find that the ground-state energy is $2E_{1,-+}^{++}(L/2, 0)$ and the constant ζ_{XY} appearing in (5.1) can be estimated from the sequence

$$\begin{aligned} Z_{L/2} &= [E_{1,-+}^{++}(L/2, 1) + E_{1,-+}^{++}(L/2, 0)] - 2E_{1,-+}^{++}(L/2, 0) \\ &= \frac{2\pi}{L}\zeta_{XY} + o(L^{-1}) \end{aligned} \tag{5.14}$$

which gives us, from (5.6), $\zeta_{XY} = 2\zeta_I$. This fact together with (5.1) implies that the conformal anomaly of the triplet XY model is double that of the triplet Ising model. Thus, the triplet XY model has for $0 < \alpha \leqslant 1$ a line of continuously varying critical exponents ($\lambda = \lambda_c = 1$) governed by a $c = 2$ conformal theory and for $\alpha > 1$ it has two phase transitions governed by a $c = 1$ conformal theory.

The scaling dimensions of the triplet XY model due to the restrictions (2.8) are double those of the triplet Ising model, except for the dimensions associated with the conformal towers belonging to the sector containing the ground state. In this last case, which includes the identity ($X=0$), the energy ($X=X_\varepsilon$), and the marginal operator ($X_{\mathrm{mar}}=2$), the dimensions in both models are the same.

6. CONCLUSION AND SUMMARY

In this paper we have introduced and studied the triplet XY quantum Hamiltonian (2.1). By making a change of variables, we have shown that this Hamiltonian is related to two alternate triplet XY Hamiltonians (3.1), each having half of the interactions and satisfying the constraints (2.8a). These alternate models have a $Z(2)$ local gauge symmetry and in Section 4 we showed that they are exactly related to the triplet Ising model[3] (4.1) with several boundary conditions imposed. For each choice of the gauge in the first model there corresponds a certain boundary condition in the second.

These facts induced us to extend (Section 5) for general boundaries the finite-size studies of the triplet Ising model already done in the periodic case.[3] Exploring the consequences of the conformal invariance of the infinite system in the finite-lattice spectrum, we calculated several scaling dimensions. Our results reveal that although in the periodic case this model shows the same anomalous dimensions[3] as the Ashkin–Teller model, both models behave differently under other boundary conditions. The parafermions of spin $S=1/4$ and spin $S=3/4$ which are present in the Ashkin–Teller model[17,21] do not occur in the triplet Ising chain with toroidal boundary conditions. For completeness, we also studied the triplet Ising Hamiltonian with free ends and we calculated the surface critical exponents.

The spectrum of the triplet XY model was obtained by combining identical sectors (same parities) of two triplet Ising models with the same boundary condition. From this combination we conclude that for $0<\alpha\leqslant 1$ the triplet XY model also has a critical line ($\lambda=\lambda_c=1$) with continuously varying critical exponents, its critical fluctuations being described by a conformal theory with central charge $c=2$. For $\alpha>1$ the model undergoes two phase transitions with fixed exponents and is governed by a $c=1$ conformal theory.

The results of Section 5 also show that the anomalous dimensions associated with eigenstates belonging to the ground-state sector are the same as the corresponding ones in the triplet Ising model. This is the case of the dimensions associated with the energy operator and the marginal operator appearing in the region where $0<\alpha\leqslant 1$. All the other dimensions which occur in the triplet Ising model with general toroidal boundary conditions appear doubled in the triplet *XY* model with periodic boundary condition.

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